Concentration of entropy of random states (arXiv:quant-ph/0407049)

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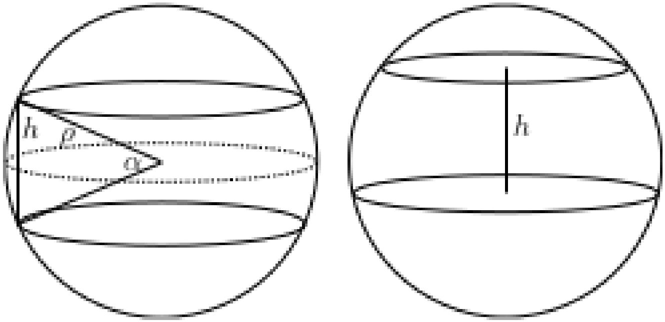
Overview

1. Concentration of Measure phenomenon
   - Measure Concentration on a Sphere
   - Levy's Lemma
2. Concentration of Entropy
   - Bounding the Lipschitz constant of $S(\varphi_A)$
   - Bound for $\mathbb{E}S(\varphi_A)$
   - Almost maximally entangled states!
   - Interpretation of concentration of entropy

Surface Area of a Strip on 2-sphere



The area covered by a strip of height of height h on a 2-dimensional sphere of radius ρ is $2\pi\rho h$

Measure concentration on $\mathbb{S}^{d-1}$

The striking fact is that for a large d, we only need small α to cover a large percentage of the surface area of the unit d-sphere (Proof awaits!).

Formally, we can define a probability measure on the whole normalized surface (with measure 1), such that for $E \subseteq \mathbb{S}^{d-1}$

$$P(E) = \frac{\sigma_{d-1}(E)}{\sigma_{d-1}(\mathbb{S}^{d-1})} \tag{1}$$

For $A \subseteq \mathbb{S}^{d-1}$, and $t > 0$, define $A_t = \{x \in \mathbb{R}^d : dist(x, A) \leq t\}$. The points in A_t are at most distance t away from A.

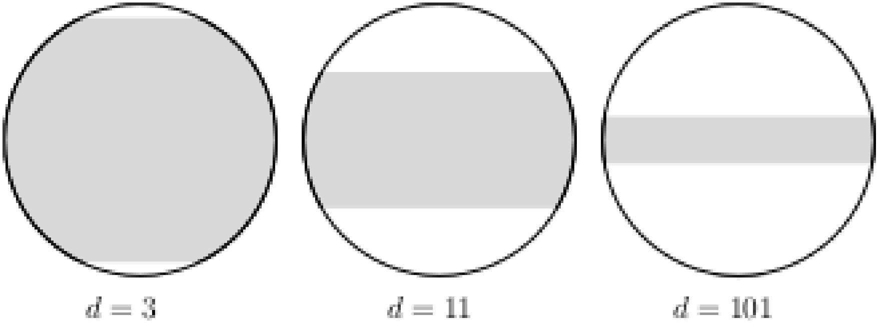
Measure concentration on $\mathbb{S}^{d-1}$

Theorem

Let $A \subseteq \mathbb{S}^{d-1}$ with $P(A) \geq \frac{1}{2}$. Then for any real $t > 0$,

$1 - P(A_t) \leq 2e^{\frac{-t^2 d}{2}}$

To obtain the strip around the equator, invoke the above by first taking A to be the the northern hemisphere and then the southern hemisphere.



Levy's Lemma

Lemma (Lemma III.1 in [1])

Let $f : \mathbb{S}^k \to \mathbb{R}$ be a function with Lipschitz constant η and a point $X \in \mathbb{S}^k$ be chosen uniformly at random. Then

$$\Pr\{f(X) - \mathbb{E}f < -\alpha\} \quad \leq \quad 2\exp\left(-C_1(k+1)\frac{\alpha^2}{\eta^2}\right)$$

where $C_1 > 0$.

Intuition: A slowly varying function on the sphere (having small η) will take values close to the average. The probability of being close to the average gets close to 1 exponentially fast with increasing k

Bounding the Lipschitz constant of $S(\varphi_A)$ (1)

We let $f(|\varphi\rangle) = S(\varphi_A)$, where $|\varphi\rangle \in A \otimes B$. Since a $d_A d_B-$dimensional quantum state can be represented by $2d_A d_B$ real vectors. Further noting that any point on a k-sphere exists in $(k+1)$-dimensional Euclidean the k in Levy's Lemma turns out to be $2d_A d_B - 1$.

To apply Levy's Lemma to the entropy of a reduced quantum state, what remains is to bound the Lipschitz constant of $S(\varphi_A)$.

Lemma (Lemma III.2 in [1])

The Lipschitz contant η of $S(\varphi_A)$ is upper bounded by $\sqrt{8}\log(d_A)$, for $d_A \geq 3$

Proof.

The strategy is to bound the Lipschitz constant of the Shannon, H, entropy of a measurement M, $H(M(\varphi_A))$ and then use that to bound $S(\varphi_A)$.

Bounding the Lipschitz constant of $S(\varphi_A)(2)$

Proof (continued).

The Schmidt decomposition of a bipartite pure state $|\varphi\rangle = \sum_{jk} \varphi_{jk} |e_j\rangle_A |f_k\rangle_B$ for some orthornormal bases $\{|e_j\rangle_A\}$ and $\{|f_k\rangle_B\}$.

Since the reduced state on A (actually on either of the two subsystems – same spectrum) will give a diagonal matrix with $|\varphi_{jk}|^2$ on the diagonals,

$$p(j|\varphi) = {}_A\langle e_j|\varphi_A|e_j\rangle_A = \sum_k |\varphi_{jk}|^2 ,$$

Then, the von Neumann entropy of the reduced state on A

$$g(\varphi) = H(M(\varphi_A)) = -\sum_j p(j|\varphi) \log p(j|\varphi) .$$

Proof (continued).

Given that the states are normalized, one can obtain

$$\eta^2 = \sup \nabla g \cdot \nabla g \;=\; \sum_{jk} \frac{4|\varphi_{jk}|^2}{(\ln 2)^2}[1 + \ln p(j|\varphi)]^2$$
$$\leq \frac{4}{(\ln 2)^2}[1 + \sum_j p(j|\varphi)(\ln p(j|\varphi))^2]$$

$\sum_j p(j|\varphi)(\ln p(j|\varphi))^2$ can be optimized with $\sum_j p(j|\varphi) = 1$ constraint using the following Lagrangian

$$\mathcal{L} = \sum_j p(j|\varphi)(\ln p(j|\varphi))^2 + \lambda(1 - \sum_j p(j|\varphi))$$
$$\forall j, \;\; \frac{\partial \mathcal{L}}{\partial p(j|\varphi)} = \ln p(j|\varphi))^2 + 2\ln p(j|\varphi)) - \lambda = 0$$

Proof (continued).

If $d_A \geq 3, \lambda = \ln p(j|\varphi))^2 + 2\ln p(j|\varphi))$ holds if $p(1|\varphi) = p(2|\varphi) = \cdots = p(d_A|\varphi) = \frac{1}{d_A}$. Then,

$$\sum_j p(j|\varphi)(\ln p(j|\varphi))^2 \leq \sum_j \frac{1}{d_A}(\ln \frac{1}{d_A})^2 = (\ln d_A)^2$$

Therefore,

$$\begin{aligned} \eta^2 &\leq \frac{4}{(\ln 2)^2}[1 + \sum_j p(j|\varphi)(\ln p(j|\varphi))^2] \\ &\leq \frac{4}{(\ln 2)^2}[1 + (\ln d_A)^2] \leq 8(\log d_A)^2, \end{aligned}$$

Choosing the measurement M along the eigenbasis of φ_A ensures $H(M(\varphi_A)) = S(\varphi_A)$. This completes the proof. $\square$

Bound for $\mathbb{E}S(\varphi_A)$

The final ingredient needed to invoke the Levy's lemma for $S(\varphi_A)$ is to obtain a bound for $\mathbb{E}S(\varphi_A)$

Lemma (Lemma II.4 in [1])

Let $|\varphi\rangle$ be chosen according to the unitarily invariant measure on a bipartite system $A \otimes B$ with local dimensions $d_A \leq d_B$. Then

$$\mathbb{E}S(\varphi_A) > \log d_A - \frac{1}{2}\beta\,,$$

where $\beta = \frac{1}{\ln 2}\frac{d_A}{d_B}$.

Almost maximally entangled states!

Theorem

Let $|\varphi\rangle$ be chosen according to the unitarily invariant measure on a bipartite system $A \otimes B$ with local dimensions $d_B \geq d_A \geq 3$. Then

$$\Pr\left\{S(\varphi_A) - \log d_A < -\alpha - \frac{1}{2}\beta\right\} \leq 2\exp\left(-\frac{d_A d_B C_3 \alpha^2}{(\log d_A)^2}\right),$$

where $C_2 = C_1/4 > 0$

Proof.

Taking $f(\varphi) = S(\varphi_A)$ in Levy's Lemma, we have
$\Pr\{f(X) - \mathbb{E}f < -\alpha\} = \Pr\{S(\varphi_A) - \mathbb{E}S(\varphi_A) < -\alpha\} \leq$
$2\exp\left(-C_1(k+1)\frac{\alpha^2}{\eta^2}\right)$ Having established that $\eta \leq \sqrt{8}\log d_A$ and
$\mathbb{E}S(\varphi_A) < \log d_A - \frac{1}{2}\beta$, we get
$\Pr\left\{S(\varphi_A) - \log d_A < -\alpha - \frac{1}{2}\beta\right\} \leq 2exp(-\frac{C_1 2 d_a d_B \alpha^2}{8(\log d_A)^2})$

□

What does the concentration of entropy mean?

It can be shown that using Lagrange multipliers (as shown earlier) that the maximum entropy of a d-dimensional is $\log d_A$

The theorem says that if you trace out ANY random pure state, the reduced subsystem is almost always maximally entangled

"almost always" is dictated by an extremely high probability that approaches unity exponentially fast with the increasing dimension of the subsystem A.

References

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Matousek, Jiri. Measure Concentration and Almost Spherical Sections, Lectures on Discrete Geometry (2006). https://www.springer.com/gp/book/9780387953731

Thank you! Questions?